Int. J. Heat Mass Transfer. Vol. 8, pp. 253–258. Pergamon Press 1965. Printed in Great Britain

LAMINAR-FLOW HEAT TRANSFER IN THE ENTRANCE REGION OF CIRCULAR TUBES

D. L. ULRICHSON* and R. A. SCHMITZ
Department of Chemistry and Chemical Engineering,
University of Illinois, Urbana, Illinois

(*Received* 21 *April* 1964 *and in revised form* 19 *August* 1964)

Abstract—Numerical solutions for heat transfer and simultaneous development of the parabolic velocity profile in circular tubes are presented for the cases of constant wall heat flux and constant wall temperature. The work is a refinement of earlier work of Kays to include the radial component of velocity in the entrance region. The radial component is obtained from Langhaar's approximate profile and the continuity equation. The results show that the effect of the radial velocity is to cause a significant decrease in the calculated local Nusselt number in the entrance region from that obtained by Kays.

NOMENCLATURE

C_p, specific heat;
I_n, modified Bessel function of order n;
k, thermal conductivity;
m, mass flow rate;
Nu, local Nusselt number;
Pr, Prandtl number;
q_w, wall heat flux;
r, ratio of radial distance to tube radius;
R, tube radius;
Re_r, Reynolds number based on tube radius;
T, temperature;
T', dimensionless fluid temperature for a constant wall heat flux;
T^*, dimensionless fluid temperature for a constant wall temperature;
u, ratio of axial velocity component to mean velocity;
v, ratio of radial velocity component to mean velocity;
x, ratio of axial distance to tube radius;
β, function defined in equation (2).

Subscripts
B, bulk mean;
0, tube inlet;
w, tube wall.

* Presently with E. I. Du Pont de Nemours and Company, Wilmington, Delaware.

INTRODUCTION

THE PROBLEM to be considered in this paper is the simultaneous development of velocity and temperature profiles for a Newtonian fluid in laminar flow in a circular tube. The main objective here is to apply the axial velocity component obtained for the entrance length by Langhaar [1] and a radial component obtained from Langhaar's profile and the continuity equation, for the purpose of investigating the effect of the radial velocity component and to present this velocity description as a useful one for the study of entrance length processes in general. In this report numerical solutions are presented for a constant wall heat flux and a constant wall temperature.

Kays [2], in the study of entrance heat transfer, and Bosworth and Ward [3], studying mass transfer in the entrance length, have used Langhaar's velocity data and neglected the radial velocity component to obtain a solution of the governing transport equations by a finite-difference method. According to boundary layer theory the radial convective term in the transport equations is not negligible, and one might expect significant errors, at least very near the inlet, if it is taken to be zero.

Other methods employed in entrance studies include series expansion and integral techniques, neither of which is readily applicable in general to more complicated processes such as those involving homogeneous chemical reaction and many others of practical significance.

Recently Heaton, Reynolds and Kays [4] employed an extension of Langhaar's technique to obtain analytically a solution for heat transfer in the entrance length with constant wall heat flux for laminar flow in concentric annuli and circular tubes. Although the method is not generally applicable, it provides a convenient source of data for cases involving a constant wall flux. The theoretical and experimental results of Heaton, Reynolds and Kays are compared in this report with the theoretical results of this work.

FLOW DEVELOPMENT IN CIRCULAR TUBES

Langhaar [1] obtained an approximate solution for the axial velocity component in the entrance region of a circular tube by linearizing the boundary-layer equation. The resulting solution is

$$u = \frac{I_0(\beta) - I_0(\beta r)}{I_2(\beta)} \tag{1}$$

where I_0 and I_2 are modified Bessel functions of orders zero and two respectively, and β is a function of x/Re_r defined by

$$x/Re_r = \int_\beta^\infty g(\beta) f'(\beta)\, \mathrm{d}\beta \tag{2}$$

where

$$g(\beta) = I_2(2\beta I_1 - \beta^2)$$

$$f(\beta) = [4I_0 I_2 - (I_0 - 1)^2 - 2I_1^2]/2I_2^2$$

$$f'(\beta) = \mathrm{d}f(\beta)/\mathrm{d}\beta$$

The argument of the modified Bessel functions is β.

To meet the objectives of this report an expression for the radial velocity component was conveniently obtained from equation (1) and the continuity equation. The continuity equation in cylindrical coordinates is

$$\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial (rv)}{\partial r} = 0$$

Thus

$$v = -\frac{1}{r}\int_0^r r\left(\frac{(\partial u)}{(\partial x)}\right)\mathrm{d}r = -\frac{1}{r}\frac{\partial}{\partial \beta}\left[\int_0^r ru\,\mathrm{d}r\right]\frac{\mathrm{d}\beta}{\mathrm{d}x} \tag{3}$$

From equation (2)

$$\frac{\mathrm{d}\beta}{\mathrm{d}x} = -\frac{1}{Re_r}\left[\frac{1}{g(\beta)f'(\beta)}\right] \tag{4}$$

From equations (1), (3) and (4) there results for the radial velocity component

$$v\,Re_r = \frac{1}{I_2\, g(\beta) f'(\beta)}\left\{\left[\frac{rI_0}{2} - \frac{I_1(\beta r)}{\beta}\right]\left(\frac{I_1}{I_2} - \frac{2}{\beta}\right) - \frac{r}{2}\left[I_1 - \frac{2I_2(\beta r)}{\beta}\right]\right\} \tag{5}$$

where the argument of the modified Bessel functions is β unless otherwise indicated. It might be noted that the computation of v from equation (5) does not constitute a major increase in time and labor since $g(\beta)$, $f'(\beta)$, I_0, I_1 and I_2 are all involved in the computation of u.

HEAT TRANSFER IN THE ENTRANCE REGION

The heat-transfer processes under investigation are governed by the boundary-layer energy equation which may be written in cylindrical coordinates as

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r} = \frac{1}{Re_r Pr}\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) \tag{6}$$

where the velocity components u and v are given by equations (1) and (5) respectively. In fully developed flow $v = 0$, and u is independent of axial distance, in which case the local mean temperature and local Nusselt number depend only on $(x/Re_r Pr)$—the well-known Graetz problem. In the entrance region, however, u depends not only on radial position but also on x/Re_r or $(x/Re_r Pr)Pr$. In addition, the transverse velocity component, v, is given in equation (5) as the product of Re_r and a function of radial position and $(x/Re_r Pr)Pr$. Thus, if the axial variable is taken as $x/(Re_r Pr$, the Prandtl number is a parameter.

In the following paragraphs numerical solutions of the finite difference form of equation (6) are presented for $Pr = 0{\cdot}7$. Solutions were obtained on the IBM 7094 digital computer for wall conditions of constant temperature and constant heat flux using thirty equally spaced radial stations with an increment size in $x/Re_r\,Pr$ of 0·001 very near the entrance and 0·01 farther downstream. Simpson's rule for integration was employed in integrations to evaluate bulk mean temperatures. In the computation of local Nusselt numbers, temperature gradients at the wall were evaluated by means of a three-point differentiation formula.

For an indication of the accuracy of the numerical solutions, some calculations were carried out using forty radial stations and an axial increment size of 0·0005. Agreement of the average temperatures and local Nusselt numbers thus obtained with those obtained using the increment sizes above was within two per cent near the inlet and within one per cent over the major portion of the entrance length.

Constant wall heat flux

For a constant wall heat flux it is convenient to introduce into equation (6) a dimensionless temperature, T', defined as

$$T' = \frac{k}{q_w R}(T - T_0) \tag{7}$$

The boundary conditions on T' are

$$\text{at } x = 0,\ T' = 0$$
$$r = 0,\ \partial T'/\partial r = 0$$
$$r = 1,\ \partial T'/\partial r = 1$$

The effect of this substitution on equation (6) is simply the replacing of T by T'.

The bulk mean temperature, T'_B, is given by

$$T'_B = 2\int_0^1 uT'r\,\mathrm{d}r \tag{8}$$

Also from an over-all heat balance up to any point x, T'_B is given by

$$T'_B = \frac{2x}{Re_r\,Pr} \tag{9}$$

independent of the velocity distribution. Equation (9) therefore provides a check on the accuracy of the numerical solution of equation (6). According to equation (9), values of T'_B computed from equation (8) should lie on a straight line of slope two when plotted versus $x/Re_r\,Pr$.

Figure 1 shows the axial variation of T'_B resulting from equations (8) and (9) and the numerical solution of equation (6) for $Pr = 0{\cdot}7$. As shown on the graph, values of T'_B from equation (8) agree with equation (9) when the transverse velocity component from equation (5) is included, but show a deviation of about − 6 per cent at the end of the hydrodynamic entrance length when v is taken to be zero.

In terms of dimensionless quantities the local Nusselt number is defined as

$$Nu = \frac{2}{T'_w - T_B} \tag{10}$$

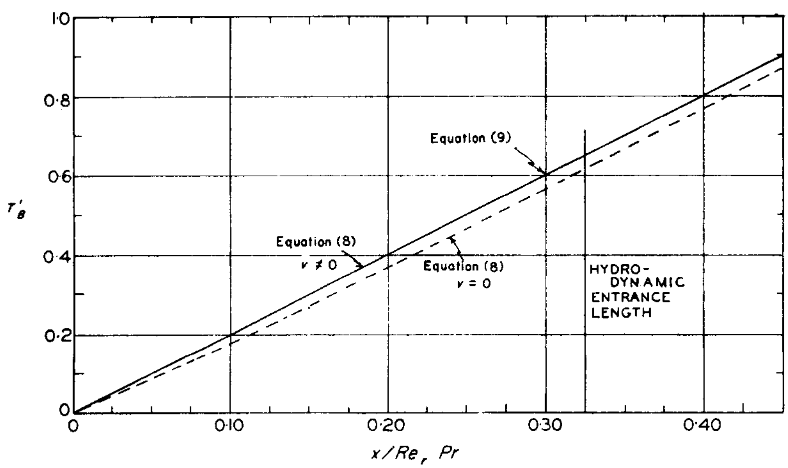

FIG. 1. Axial variation of bulk temperature for constant wall flux and $Pr = 0{\cdot}7$.

Figure 2 shows the local Nusselt number resulting from the numerical solutions and compares the result obtained when the radial velocity is retained to those obtained when it is neglected. The curve of Fig. 2 for $v = 0$ gives values somewhat different from those presented in the original work of Kays [2] since the larger number of radial increments used in this work has improved the accuracy considerably. The results show that the local Nusselt number near the entrance is overestimated by about 6 per cent if the radial velocity is neglected. The fact that the local Nusselt number is lower when the radial velocity is not neglected seems at first to be contradictory to reason. This phenomenon, however, is readily explained as being the result of the fact that the flowing fluid is not actually being heated by the flux of heat specified at the wall when v is taken to be zero. This was clearly shown in Fig. 1. Thus if q_w were replaced in equation (10) by the actually calculated local rate of heating given by $mC_p(dT_B/dx)$, the resulting Nusselt number for $v = 0$ would be, as expected, below that resulting when $v \neq 0$. In the region of Fig. 1 near the entrance where the radial velocity is not negligible, $dT_B/d(x/Re_r Pr) < 2$. Through this region an appreciable error results in the local Nusselt number. The error in the local Nusselt number becomes negligible before the end of the entrance length as the error in T_B' approaches a constant value.

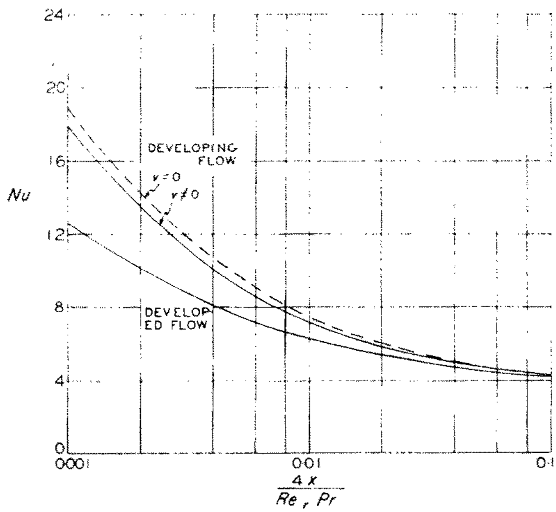

FIG. 2. Local Nusselt number for constant wall flux and $Pr = 0.7$.

Figure 3 compares the results obtained from this work for $v \neq 0$ with theoretical and experimental results of Heaton, Reynolds and Kays [4].

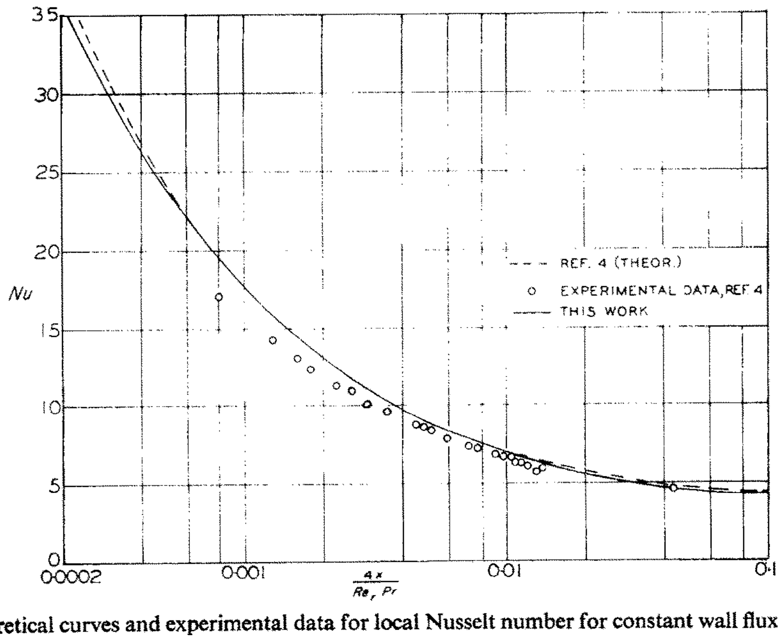

FIG. 3. Theoretical curves and experimental data for local Nusselt number for constant wall flux and $Pr = 0.7$.

Their theoretical results were obtained by an integral method which accounts for the effect of the radial velocity on the average. The theoretical curves are in very good agreement, and the results of this work give slightly better agreement with measured values, particularly near the inlet.

Constant wall temperature

In this case a dimensionless temperature, T^*, to be introduced into equation (6) is defined by

$$T^* = \frac{T - T_0}{T_w - T_0} \tag{11}$$

The boundary conditions then are

$$x = 0, T^* = 0$$

$$r = 0, \frac{\partial T^*}{\partial r} = 0$$

$$r = 1, T^* = 1$$

The local Nusselt number is given by

$$Nu = \frac{2(\partial T^*/\partial r)_{r=1}}{1 - T_B^*} \tag{12}$$

where

$$T_B^* = 2 \int_0^1 u' T^* r \, dr$$

The Nusselt number resulting from the calculations under these conditions is shown in Fig. 4. When the radial velocity is neglected the same qualitative effect is found as in the previous case of constant wall heat flux—that of overestimating the local Nusselt number. This again is due to the fact that the heat flux at the wall characterized by $(dT^*/dr)_{r=1}$ in equation (12) is considerably larger than the actual local rate of heating due to the fact that the continuity equation is not satisfied locally. The curves of Fig. 4 show that the relative error in the local Nusselt number near the inlet may be larger than 15 per cent when v is taken to be zero.

CONCLUSIONS

The velocity profile of Langhaar together with a radial component obtained through the continuity equation afford a convenient and suitably accurate description of the entrance velocity field. This description is applicable to laminar heat- and mass-transport processes in the entrance region in general when the velocity field can be assumed to be independent of the transfer processes occurring.

This work has shown that significant errors may result when the radial component of velocity is neglected and has thus provided a more accurate prediction of the entrance length heat transfer in a tube for a Prandtl number of 0·7 than has previously been reported.

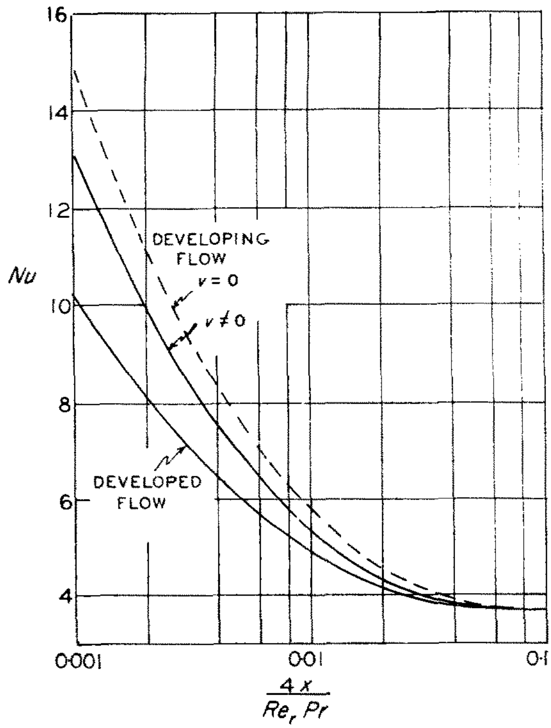

FIG. 4. Local Nusselt number for constant wall temperature and $Pr = 0{\cdot}7$.

ACKNOWLEDGEMENT

Fellowship support for one of the authors from the Sun Oil Company is acknowledged.

REFERENCES

1. H. L. LANGHAAR, Steady flow in the transition length of a straight tube, *J. Appl. Mech.* **A9**, 55–58 (1942).
2. W. M. KAYS, Numerical solutions for laminar-flow heat transfer in circular tubes, *Trans. Amer. Soc. Mech. Engrs* **77**, 1265–1274 (1955).
3. R. T. BOSWORTH and H. C. WARD, Mass transfer in

the entrance length of a pipe-laminar flow, paper No. 169, presented at the Annual Meeting of the Amer. Inst. Chem. Engrs, Chicago, Illinois (December 1962).
4. H. S. HEATON, W. C. REYNOLDS and W. M. KAYS, Heat transfer with laminar flow in concentric annuli with constant heat flux and simultaneously developing velocity and temperature distributions, Rep. No. *AHT*-5, Thermosciences Division, Dept. of Mech. Engng, Stanford University, Stanford, California (July 1962).

Zusammenfassung—Für die Fälle konstanter Wärmestromdichte an der Wand und konstanter Wandtemperatur werden numerische Lösungen für den Wärmeübergang und die gleichzeitige Ausbildung des parabolischen Geschwindigkeitsprofiles in Rohren mit Kreisquerschnitt angegeben. Diese Arbeit ist eine Verfeinerung der früheren Arbeit von Kays, indem die radiale Geschwindigkeitskomponente im Bereich des Einlaufs mit erfasst wird. Die Radialkomponente erhält man über das Näherungsprofil Langhaars und die Kontinuitätsgleichung. Die Ergebnisse zeigen, dass der Einfluss der Radialgeschwindigkeit eine bedeutende Verringerung der hier berechneten, lokalen Nusseltzahl im Einlaufsbereich gegenüber der von Kays ermittelten verursacht.

Аннотация—Приводятся численные решения задачи о теплообмене и одновременном развитии параболического профиля скорости в трубах круглого сечения для случаев постоянного теплового потока от стенки и постоянной температуры стенки. Данная работа представляет собой развитие более ранней работы Кейса так, чтобы учесть и радиальную составляющую скорости на входном участке. Эта радиальная составляющая получена по приближенному профилю Лангхаара и уравнению неразрывности. Результаты работы показывают, что радиальная скорость дает более значительное уменьшение расчетного локального числа Нуссельта на входном участке, чем это получено Кейсом.